Defect Formation Rates in Cosmological First-Order Phase Transitions

Matthew Lilley*and Antonio Ferrera†
* *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom*
* *Institut für Theoretische Physik der Universität Heidelberg Philosophenweg 16, D-69120 Heidelberg, Germany*

In cosmological first-order phase transitions, the progress of true-vacuum bubbles is expected to be significantly retarded by the interaction between the bubble wall and the hot plasma. It has been claimed that this leads to a significant reduction in the number of topological defects formed per bubble, as a result of phase equilibration between bubbles. This claim has been verified for spontaneously-broken global symmetries. We perform a series of simulations of complete phase transitions in the $2+1$-dimensional $U(1)$-Abelian Higgs model, for a range of bubble wall velocities, in order to obtain a quantitative measure of the effect of bubble wall speed on the number density of topological defects. We find that the number of defects formed is i) significantly lower in the local than the global case and ii) decreases exponentially as a function of wall velocity. Slow-moving bubbles also lead, however, to the nucleation of more bubbles before the phase transition is complete. Our simulations show that this is in fact the dominant effect, and so we predict *more* defects per unit volume as a result of the sub-luminal bubble wall terminal velocity.

PACS numbers: 98.80.Cq, 11.27.+d, 64.60.Qb, 64.60.-i

I. INTRODUCTION

As the early Universe expanded and cooled, it is expected to have undergone a series of phase transitions at each of which a symmetry which had been thermally-restored became spontaneously-broken. At each of these phase transitions, depending on the nature of the symmetry-breaking involved, topological defects may or may not have formed [1]. Phase transitions are labelled first- or second-order according to whether the field-space location of the true vacuum state changes discontinuously or continuously as the critical temperature is crossed. First-order phase transitions, with which this work is concerned, proceed by bubble nucleation and expansion. Depending on the details of the theory, when at least $(4-n)$ bubbles collide, an $n-$dimensional topological defect may form in the region between them. In this paper, we simulate a series of complete cosmological first-order phase transitions in order to quantify the initial density of topological defects.

We take as our model the simplest spontaneously-broken gauge symmetry: the Abelian Higgs model, which has a local $U(1)$ symmetry, with Lagrangian

$$\mathcal{L} = (D_\mu\Phi)^\dagger(D_\mu\Phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\Phi^\dagger\Phi), \qquad (1)$$

where $D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The detailed form of the effective potential $V(|\Phi|)$ will depend upon the particular particle-physics model being considered, but in order to be able to study the generic features of a first-order phase transition we shall take V, following [2] and [3], to be

$$V(\Phi) = \lambda\Big[\frac{|\Phi|^2}{2}(|\Phi|-\eta)^2 - \frac{\varepsilon}{3}\eta|\Phi|^3\Big], \qquad (2)$$

where in a realistic model, $\varepsilon = \varepsilon(T) \propto (T_c - T)$. The potential V has a local minimum false-vacuum state at $\Phi = 0$ which is invariant under the $U(1)$ symmetry, and global minima true-vacuum states on the circle $|\Phi| = \rho_{tv} \equiv (\eta/4)\,(3+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})$ which possess no symmetry. The dimensionless parameter ε is responsible for lifting the degeneracy between the two sets of minima – the greater ε, the greater the potential difference between the false- and true-vacuum states, and hence the faster the bubbles will accelerate. This model admits string-solutions, or in the $2+1$-dimensional case considered here, vortices.

By making the field and coordinate transformations

$$\Phi \longrightarrow \phi = \eta\Phi \qquad (3)$$

$$\mathbf{x} \longrightarrow \mathbf{x}' = \frac{\mathbf{x}}{\sqrt{\lambda}\eta} \qquad (4)$$

$$t \longrightarrow t' = \frac{t}{\sqrt{\lambda}\eta} \qquad (5)$$

it is possible to set λ and η to unity, so that the potential is parametrised only by ε, and hereafter we shall use these transformed variables.

The bubble nucleation rate per unit time per unit volume and the field-space profile of the bubble wall are obtained from the 'bounce' (i.e. least action) solution of the Euclidean action [4]. In $d+1$-dimensions the modulus ρ of the field Φ is given by the solution of the equation

$$\frac{d^2\rho}{dr^2} + \left(\frac{d}{r}\right)\frac{d\rho}{dr} - V'(\rho) = 0, \qquad (6)$$

with boundary conditions

*Electronic address: lilley@thphys.uni-heidelberg.de
†Electronic address: ceefp44@pinar2.csic.es

$$\lim_{r\to\infty} \rho(r) = 0 \quad (7)$$

$$\left.\frac{d\rho(r)}{dr}\right|_{r=0} = 0. \quad (8)$$

Ignoring quantum fluctuations, the phase θ is constant within each bubble, and uncorrelated between spatially-separated bubbles. Any non-zero gauge fields in the nucleation configuration will make a contribution to the action and hence the nucleation of bubbles with non-zero gauge fields is exponentially suppressed. When three or more bubbles collide, a phase-winding of $2\pi n$ can occur around a point, which by continuity must then be at $\Phi = 0$. In three spatial dimensions, this topologically-stable region of high-energy false vacuum is string-like – a cosmic string. If the phases of the bubbles were uniformly-distributed, it is simple to calculate that at every three-bubble collision, there is a probability $p = 0.25$ that the arrangement of phases would be such as to generate the winding required to form a defect. We will use this value of $p = 0.25$ as our benchmark throughout this work. This description of defect formation and the estimation of the initial defect density is known as the Kibble mechanism, and the purpose of this work is to investigate to what extent its predictions are modified by the early-Universe environment in which cosmological phase transitions take place.

Although two-dimensional (or domain walls) [5] and pointlike defects (monopoles) [6,7] would dominate the energy density of the Universe, one-dimensional ‘cosmic strings’ are expected to reach a scaling solution (where the ratio of the string energy density to that of the background remains constant), and thus need not be pathological. For many years, cosmic string-seeded perturbations were considered a viable candidate for the primordial density fluctuations which led to the formation of structure in the Universe. The failure of cosmic string-induced CMB anisotropy spectra [8] to match observations [9,10] however, has laid rest to this theory. This does not mean though that cosmic strings did not form, or that if they did, they are cosmologically-irrelevant.

Cosmic strings may be either closed loops, or infinitely-long strings (see [11] or [12] for cosmic string reviews). Closed loops will decay via gravitational radiation, forming a gravitational-wave background which could be detectable in the planned experiments LIGO and LISA. Cosmic strings have been cited as being responsible for baryogenesis [13,14], if they trap a Grand Unified gauge field with baryon-number violating interactions in their core. They could cause the highest energy cosmic rays [15], whose existence has largely defied explanation – standard methods of ray production such as quasars do not appear to be able to generate such events. When string loops double back on themselves, forming a ‘cusp’, a jet of ultra-high energy accelerated particles is emitted, which could explain the origin of Gamma Ray Bursts [16]. It has even been proposed that cosmic strings may constitute the dark matter [17] – a network of non-Abelian (in order to prevent a scaling solution being reached) strings would have negative pressure and could therefore explain the why the Universe appears to be accelerating.

In order to be able to assess the significance of cosmic strings in the evolution of the early Universe, it is important to be able to estimate the initial defect density accurately. If the phase transition at which they formed was first-order, this would depend on how the phases between two or more bubbles interpolate after collision. In particular, although strings are in general formed when three or more bubbles collide, a simultaneous three-bubble collision is unlikely – one would expect in general two-bubble collisions, with a third, or fourth bubble colliding some finite time later. If the phase inside a two-bubble collision is able to equilibrate quickly, and before a third bubble arrives, there may be a strong suppression of the initial string density. The effect of phase equilibration on the initial defect density was first investigated by Melfo and Perivolaropoulos [18]. They found a decrease of less than 10% (i.e. $0.22 < p < 0.25$), in models which possess a global symmetry and with bubbles which, upon nucleation, accelerated up to the speed of light.

The above description of defect formation, however, ignores any effect that the hot-plasma background may have on the evolution of the Higgs field, which may be significant in the early Universe. Real-time simulations [19] and analytic calculations [20] for the (Standard Model) electroweak phase transition predicted that the bubble wall would reach a terminal velocity $v_{\rm ter} \sim 0.1c$. Recent calculations for the Minimal Supersymmetric Standard Model, where there are many more particles, and hence more interactions and hence a more viscous plasma, show that the bubble wall velocity could be as low as $v_{\rm ter} \sim 10^{-3}c$ [21]. The reason for this is simple: outside the bubble, where the $(SU(2)\times U(1))$ symmetry remains unbroken, all fields coupled to the Higgs are massless, acquiring their mass from the vacuum expectation value of the Higgs in the spontaneously-broken symmetry phase inside the bubble. Particles outside the bubble without enough energy to become massive inside bounce off the bubble wall, retarding its progress through the plasma. The faster the bubble is moving, the greater the momentum transfer in each collision, and hence the stronger the retarding force. Thus a force proportional to the bubble-wall velocity appears in the effective equations of motion.

Ferrera and Melfo [2] studied bubble collisions in such an environment, for theories which possessed a global symmetry, and found that decaying phase oscillations occur inside a two-bubble collision, which was claimed would lead to a suppression of the defect formation rate. Kibble and Vilenkin [22] studied phase dynamics in collisions of undamped bubbles in models with a local symmetry, and found, analytically, a different kind of decaying phase oscillation. When the finite conductivity of the plasma was included, these oscillations were found not to occur. Davis and Lilley [3] extended this work, to include slow-moving bubbles in local-symmetry models. Simulating the collision and merging of two bubbles, it was

found that the phase oscillations were suppressed – the (gauge-invariant) phase difference between two bubbles equilibrated very rapidly upon collision (and even more rapidly when the conductivity of the plasma was taken into account), with the implication that there would be a suppression of defect formation in slow-moving bubbles in gauge theories as well. This claim was illustrated by an example of an undamped 3-bubble collision which led to the formation of a defect, but when identical initial conditions were evolved in a damped environment, no defect formed.

Ferrera [23] confirmed the hypothesis of [2] by performing a series of simulations of complete global-symmetry phase transitions at different values of the bubble wall terminal velocity. However, quantitative analysis of the defect formation rate in the most realistic scenario cosmologically – a gauge-theory phase transition where the bubbles are slowed significantly by the plasma (as might be expected at the electroweak- or GUT-scales) – has not been studied.

In this paper we present the results of our investigations into the effect of the sub-luminal bubble wall velocity in gauge theory phase transitions. We have simulated a series of complete phase transitions, from the nucleation of the first bubble in an entirely symmetric background until $> 95\%$ of the available space having been converted into the broken-symmetry phase. We have done this (as in [23], but in a local rather than global symmetry) for a range of values of the bubble wall velocity in order to obtain a qualitative measure of the dependence of n_d on $v_{\rm ter}$. Our results (which are displayed in Figure 5) show that the two hypotheses of [3] *do* hold: for any value of the bubble wall speed there are fewer defects in gauge theory phase transitions than in global theory ones, and the number of defects formed per bubble decreases with decreasing wall velocity. For values of the wall velocity which are high compared with those calculated for cosmological phase transitions, we find already almost an order of magnitude fewer defects. We also confirm the gauge-theory analogue of the result of [18] – if the bubbles move at the speed of light, there is no significant reduction in the defect formation rate due to phase equilibration.

As well as the possibility of phase equilibration and the suppression of defect formation however, slow-moving bubbles will also require a larger number of bubbles to be nucleated in order to complete the phase transition. These two effects act in opposition, and our simulations show, for the range of wall velocities considered, that the former is dominant, i.e. the net effect of slow-moving bubbles is that *more* defects per unit volume will be formed than one would expect were the bubbles to move at the speed of light.

We should note in passing that we have ignored the effect of the expansion of the Universe in our work. This is a good approximation for phase transitions which take place at late times, like the electroweak phase transition. At phase transitions which occur earlier, however, the Hubble expansion may have a significant effect on bubble and phase dynamics. This topic deserves consideration on its own.

II. SIMULATING A PHASE TRANSITION

In the previous section we have described how the sub-luminal terminal velocity of the bubble walls causes the gauge-invariant phase difference between two bubbles to equilibrate more quickly than in the undamped case. It has been demonstrated, qualitatively, how this can lead to the suppression of topological defect formation [3], and we have explained how this could be cosmologically significant. In order to set this work on firmer footing, and be in a position to assess the cosmological consequences, we would like now to be able to quantify these effects.

In particular, we would like to be able to investigate two of the claims made in [3]:

1. For a given wall velocity, fewer topological defects are formed in local-symmetry models than in those with a global symmetry.

2. In models where a local symmetry is spontaneously broken, the average number of defects formed per bubble nucleated is a decreasing function of the bubble wall velocity.

In order then to quantify the effect on defect formation probabilities of the slow-moving bubble walls, we must perform many simulations of 'complete' phase transitions each involving the nucleation of many bubbles. Such an investigation was carried out for values of the terminal velocity in the range $v_{\rm ter} = c$ to $v_{\rm ter} \approx 0.05c$. This paper describes the procedure followed and presents the results obtained.

In the standard [25] computer simulations of defect formation, causally-disconnected points on the spatial lattice are assigned randomly-generated relative phases (corresponding to the centres of true-vacuum bubbles in a first-order phase transition, or of domains in a second-order transition). The phase between sites is then taken to vary on the shortest path on the vacuum manifold – the so-called geodesic rule. Topological defects will then form wherever this geodesic interpolation between sites generates a topologically-nontrivial path in the vacuum manifold. For a first-order transition this formalism corresponds to true vacuum bubbles nucleating simultaneously, equidistant from all their nearest neighbors. Consequently all collisions between neighboring bubbles occur simultaneously, and the associated phase differences are simply given by the differences in the initial assigned phases.

We, on the other hand, wish to investigate precisely the effect on the defect formation probability of phase equilibration – something which is only possible as a result of the non-simultaneity of three-bubble collisions.

The procedure described above was formulated as a consequence of the limitations of computing power available at the time. Now, however, with the increased processor capabilities, we have the opportunity to perform much more computationally-intensive simulations. As a result, instead of following the standard procedure, we have used a spatial lattice several times smaller than the width of the bubble wall, and evolve the discretised field equations in their entirety. This is necessary as it is the dynamics of the phase and gauge fields which lead to the suppression of defect formation seen in [3].

III. PHASE TRANSITION ALGORITHM

We wish to simulate a first-order phase transition in the Abelian-Higgs Model, which possesses a $U(1)$ gauge symmetry and has equations of motion

$$\ddot{\rho} - \rho'' - (\partial_\mu \theta - eA_\mu)^2 \rho = -\frac{\partial V}{\partial \rho} \tag{9}$$

$$\partial^\mu \left[\rho^2 (\partial_\mu \theta - eA_\mu)\right] = 0 \tag{10}$$

$$\ddot{A_\nu} - A_\nu'' - \partial_\nu \left(\partial \cdot A\right) = -2e\rho^2 \partial_\nu \theta, \tag{11}$$

where we will take the potential V to be given by

$$V(\Phi) = \lambda \Big[\frac{|\Phi|^2}{2}(|\Phi| - \eta)^2 - \frac{\varepsilon}{3}\eta|\Phi|^3\Big], \tag{12}$$

with λ and η re-scaled to unity in the manner described in the Introduction. As we have stated however, we would like to investigate the behaviour of the phase in collisions of slow-moving bubbles. For a given theory, by considering the Boltzmann equations for scattering off the Higgs field, it is possible to calculate the terminal velocity of the bubble wall [31]. Since we are not concerned here with the parameters of a specific particle-physics model, we choose instead to use a single damping parameter Γ to model the interaction of the Higgs with the plasma. In the introduction we claimed that the plasma would introduce a term proportional to the bubble-wall velocity into the equations of motion. Since the phase θ of the Higgs field is not affected by the effects described, we assume that the plasma couples only to the modulus ρ. We then have an effective equation of motion for ρ

$$\ddot{\rho} - \rho'' + \Gamma\dot{\rho} - (\partial_\mu \theta - eA_\mu)^2 \rho = -\frac{\partial V}{\partial \rho}, \tag{13}$$

replacing equation (9). Equations (10) and (11) remain unchanged.

A damping term of this form has been used by several authors [2], [30], [31], and has also been derived from the stress-energy of the Higgs, assuming a coupling to the plasma [32]. Heckler [30] estimates $\Gamma \sim g_W^2 T_c$ for the electroweak phase transition, by comparing the energy generated by the frictional damping with the pressure on the wall due to the damping.

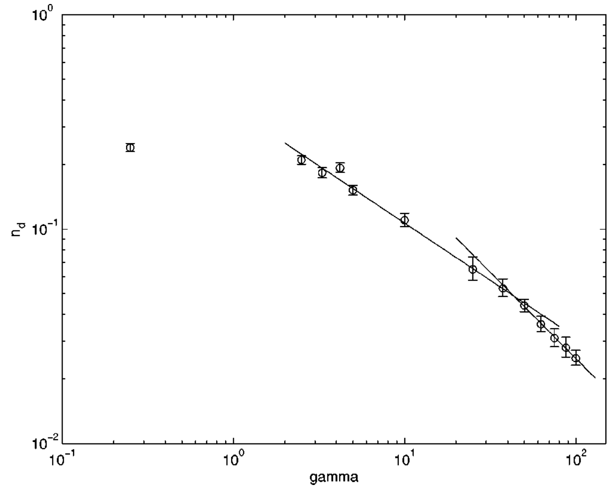

FIG. 1. Log-log plot of the number of defects formed per nucleated bubble, n_d, vs the friction coefficient gamma (from [23]). The two solid lines correspond to the least squares best fits for the two different regimes $n_d \propto \gamma^{-\alpha_{1,2}}$. The "low" friction regime has a slope of $\alpha_1 = 0.53 \pm 0.05$, whereas in the high friction regime n_d decays with $\alpha_2 = 0.8 \pm 0.3$.

The effect of this damping term is that instead of accelerating up to the speed of light, the bubble walls reach a terminal velocity $v_{\rm ter} < c$. An exact relation between Γ and $v_{\rm ter}$ can be calculated [3], but the approximation $v_{\rm ter} \sim 1/\Gamma$ is sufficiently accurate for our purposes.

The results of Ferrera [23] are shown in Figure 1 – the number of defects formed per bubble nucleated, n_d, decreases exponentially with decreasing bubble wall velocity. For terminal velocities $v_{\rm ter} \approx 0.01c$, a decrease of over an order of magnitude in the defect formation rate (i.e. $p < 0.03$) was measured.

We will then follow a modified version of Ferrera's program (as in his work using only 2+1 dimensions, again because of computational constraints – the gauge theory has many more degrees of freedom than the global theory and thus requires significantly more memory and computer time) of nucleating bubbles at random points in space and time during the course of the simulation. Our algorithm is as follows:

1. for the specific parameters of the potential under investigation, compute the bubble wall profile (numerically) from equation (6).

2. generate a population of time ordered bubble nucleation events distributed randomly within some finite volume of 2+1 dimensional space-time, assigning a random phase to each bubble.

3. start with an initial state for which the field is in the false vacuum (i.e $\Phi = 0$) in all the simulation volume, and, assuming that there is no primordial magnetic field, the gauge fields are everywhere zero.

4. at the beginning of each time step check whether there are any bubbles to be nucleated at that time:

   (a) if there are any bubble nucleation events, nucleate only those bubbles that would fall entirely in a false vacuum region and discard the rest (i.e., avoid superimposing new bubbles on regions that are already in the true vacuum).

5. evolve the resulting field configuration to the next time step, following the field equations (13), (10) and (11) (discretised in the gauge-invariant way described in, for example [33]), using a Runge-Kutta fourth-order algorithm.

In order to minimise boundary effects we nucleated bubbles only within a box which was a few bubble radii smaller than our simulation volume, as was done by Srivastava [29], and also used this smaller box to determine when the phase transition was complete.

To generate the bubble nucleation events, we first fix their number, then choose at random the space-time points at which they take place within the simulation volume. Since we only nucleate those bubbles that fall within the false vacuum and discard the rest, at later times into the transition it will become increasingly difficult for the new bubbles to find themselves in the false vacuum, and consequently fewer and fewer bubbles will be nucleated. Also, and although in a realistic situation one would expect the nucleation rate to vary with the amount of dissipation present in the system, we kept the nucleation rate constant throughout the simulation series. This is because we are concerned primarily with investigating the number of vortices produced per nucleated bubble, n_d – since computational resources are finite, we have chosen not to explore the effect of varying the nucleation rate at this stage.

The only constraint on the nucleation rate is that we wish to ensure that on average the bubbles reach their terminal velocity before colliding. The bubbles collide, on average, when their radius is of the order of the initial spatial separation of nucleation events

$$t_{\mathrm{coll}} = t(r = n_d^{1/(d+1)}), \tag{14}$$

where d is the number of spatial dimensions.

We can parametrise the radius at a given time by three quantities: the initial radius r_0, the terminal velocity v_{ter} and the time taken to reach terminal velocity t_{ter}. These three parameters are easily obtained from simulations. Assuming that the bubble wall accelerates uniformly up to v_{ter} and then moves with constant velocity, we get

$$r(t) = r_0 + v_{\mathrm{ter}}\left(t - \frac{t_{\mathrm{ter}}}{2}\right). \tag{15}$$

Then the condition $t_{\mathrm{coll}} > t_{\mathrm{ter}}$ can be expressed

$$n_d > \left[\frac{v_{\mathrm{ter}}\, t_{\mathrm{ter}}}{2} + r_0\right]^{d+1}, \tag{16}$$

and so the value of n_d used was chosen to satisfy this constraint for all values of Γ.

Similarly, we have chosen not to investigate the effect of crossing the Bogomol'nyi limit – at the critical coupling $\beta = \lambda/2e^2 = 1$, a pair of vortices remain stationary (see, for instance [11]). Above this limit they repel one another, and below they attract. We have chosen, for reasons described in section IV, parameters corresponding to $\beta < 1$ for our simulations.

IV. CHOICE OF SIMULATION PARAMETERS

The intention of this project is to investigate the effect of bubble terminal velocity on the defect formation process. It is therefore necessary to ensure that the bubbles, once nucleated, reach their terminal velocity v_{ter} as rapidly as possible in order to decrease the probability of bubbles colliding at speeds lower than v_{ter} and contaminating our results. This is achieved by choosing a relatively high value of the asymmetry parameter, $\varepsilon = 0.8$ – Figure 2 shows the effect of ε on the acceleration of a bubble. This choice also has two advantages: the ratio of the radius of a nucleated critical bubble to its thickness is relatively small (compared to lower values of ε), which aids numerical accuracy (by decreasing the spatial resolution required to produce stable results) and it is the value chosen by Ferrera for his simulations, thus enabling a direct comparison of the results in global and local-symmetry phase transitions. Figure 3 shows the difference in the bubble wall profile for small and large values of ε.

For the gauge coupling constant, we took $e = 0.25$ throughout. At each value of Γ a number of phase transitions at $e = 0$ (i.e a global symmetry model) were simulated to ensure consistency with the results of Ferrera (Figure 1). Since we primarily wish to investigate the effect of the bubble wall speed on the number of defects formed, we have chosen to perform all our calculations with the conductivity $\sigma = 0$.

V. ENDING THE PHASE TRANSITION

Vortices have dynamics, and as such it is important that we specify *when* we intend to count them. If the simulation were allowed to run indefinitely, all vortices formed would eventually annihilate, leaving none – magnetic flux is conserved, and since we started out with zero total flux, at any given time there must be an equal number of vortices and antivortices. Following [23], we have decided to terminate the simulation when 95% of the simulation volume had been converted to true vacuum. As a check, the number of vortices and antivortices present was calculated at regular intervals and stored to ensure that a) the number of defects was an increasing function of time, and b) the number of defects had remained

constant for some time before the phase transition was stopped. Any uncertainty introduced by this choice of 95% completion would have been compensated by the large number of simulations performed for each value of Γ. Indeed a glance at the cumulative vortex count shows that the number of defects present in the simulation volume does not change significantly if the truncation time is increased or decreased by 10%.

At the end of the phase transition, the number of vortices and antivortices produced were counted, using an algorithm which went across the grid checking at each point for gauge-invariant phase windings [34] of $2\pi n$ around the smallest square going clockwise from that point with all four corners in true vacuum. That the number of vortices and antivortices must be equal is another check on the accuracy of the code.

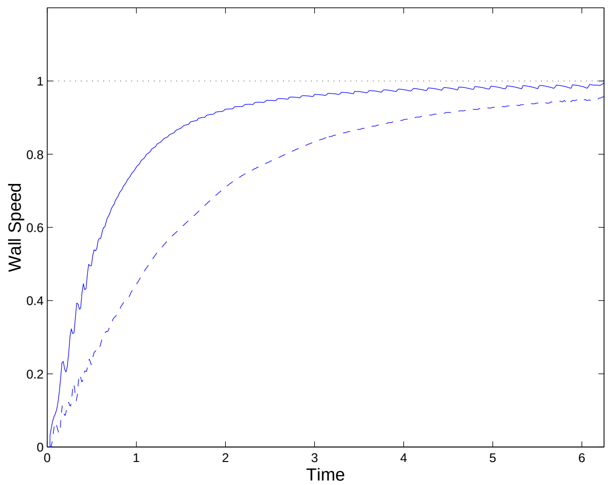

FIG. 2. Illustration of effect of asymmetry parameter ε on the acceleration of the bubble wall. The solid line shows the evolution of wall speed with time for $\varepsilon = 0.4$. This bubble, with a small radius and large difference in energy density across its membrane, accelerates rapidly towards the speed of light. The dashed line shows the wall speed for $\varepsilon = 0.1$. The speed of this bubble wall approaches unity much more slowly.

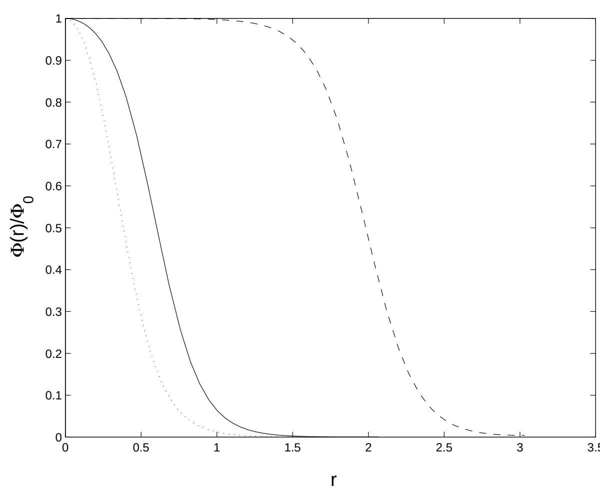

FIG. 3. Comparison of wall profiles for different parameters. The dashed line shows the profile for $\varepsilon = 0.1$, which has a thin wall and large radius. The solid line shows the profile for $\varepsilon = 0.4$ and the dotted line that for $\varepsilon = 0.8$, the value used in this paper. The radii of the bubbles decreases, and the thickness of the wall increases with increasing ε, making high values of ε most suitable for these simulations.

VI. DETAILS OF COMPUTATION

The first-order phase transition was simulated for 7 values of Γ. For $\Gamma = 0, 2, 3, 5$ and 7.5, 30 simulations were performed at each value, whereas for $\Gamma = 10$, only 5 were performed, due to the increasing amount of time taken to complete the phase transition. At each value of Γ, the simulations were run on a smaller and smaller lattice spacing with identical initial conditions, until no divergence in the results could be detected.

A lattice spacing of 0.02 was determined to be sufficient for all values of Γ and so was used throughout. For reasons of simulation time, rather than memory, the spatial lattice size was then set at 3000^2 grid points, giving a total physical area of 60^2. This box size corresponds to around 20 bubble radii, for the case where $\varepsilon = 0.8$. The time step Δt was set so that it satisfied the Courant condition (see, for instance p.829 of [26] for further details) for numerical stability – namely

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{\sqrt{d}}, \tag{17}$$

where d is the number of spatial dimensions. Altogether, we estimate that this project required around 1300 hours of combined CPU time on the supercomputer.

VII. SIMULATION RESULTS

The main result of our simulations can be seen in Figure 5. Our conclusions are twofold and can be stated thus – the number of topological defects formed per bubble nucleated *did* decrease significantly as the terminal velocity of the wall decreased; the number of topological defects formed per bubble nucleated *was* significantly lower for gauge theories than for global theories. Both of these statements support the hypotheses made in [3].

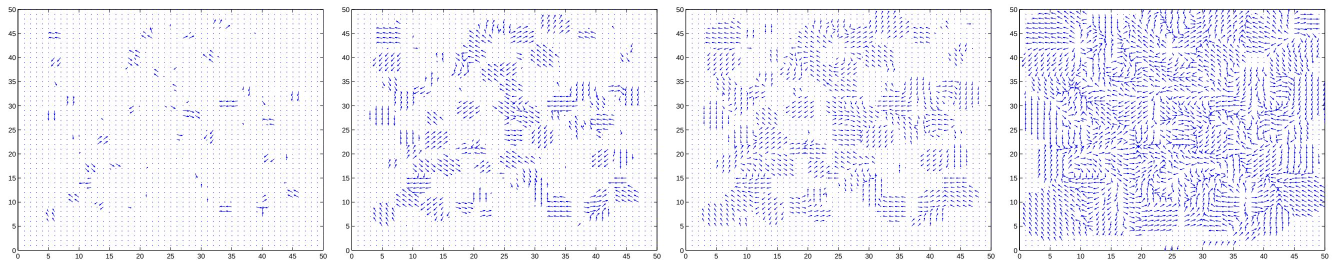

FIG. 4. Snapshots of a phase transition with $\Gamma = 0$ at different times. From top left, they are at $t = 2.2, 3.67, 5.13, 7.33$, and the phase transition was declared complete by the program at $t = 7.80$. The length of the arrows is proportional to the value of $|\Phi| = \rho$, and their direction gives the phase θ of the Higgs field. In particular, the false vacuum (which is $\Phi = 0$) corresponds to an absence of lines. The first image is very early in the phase transition, when only very few bubbles have been nucleated. These bubbles expand, alongside new bubbles which continue to be nucleated. In the second and third of these images, the existence at a given instant of bubbles of differing sizes is evident. The final image shows the phase transition almost complete.

We have shown that the number of defects formed per bubble nucleated, n_d, decreases exponentially with bubble wall velocity (since $v_{\rm ter} \sim 1/\Gamma$). The cosmologically-relevant quantity would be, of course, the number of defects per unit volume, n_v. These are related by

$$n_v = \frac{n_d}{\text{number of bubbles per unit volume}} \sim \frac{n_d}{r^{-3}}, \quad (18)$$

where r is the average bubble radius on collision. Since the only parameters in this system are the damping coefficient Γ and the bubble nucleation rate per unit time per unit volume N, we have

$$r \sim [N\Gamma]^{-\frac{1}{d+1}}, \quad (19)$$

where d is the number of spatial dimensions. If, as we have discovered, $n_d \sim \Gamma^{-\alpha}$, we have

$$n_v \sim N^{\frac{3}{d+1}}\ \Gamma^{\left(\frac{3}{d+1}-\alpha\right)}. \quad (20)$$

If we assume a fixed nucleation rate, slow-moving bubbles have two effects with respect to the formation of topological defects. The first is that more bubbles must be nucleated to complete the phase transition, and the second is that phase equilibration suppresses the formation of defects in some cases. Our results indicate, since we have found $\alpha < 1$ for $d = 2$, that the increase in the number of bubbles nucleated outstrips the decrease in the defect-formation rate due to phase equilibration, and that slow-moving bubbles lead to *more* defects in any given volume.

Here we must point out that this statement is only valid for the range of wall velocities we have considered ($0.05 \lesssim v_{\rm ter}/c \leq 1$). In this range, we have measured the exponent $\alpha_{\rm local} = 0.65 \pm 0.04$, whereas in the global case [23] $\alpha_{\rm global} = 0.53 \pm 0.05$ was found for the corresponding region. For lower terminal velocities, however, another regime was discovered, with $\alpha_{\rm global} = 0.8 \pm 0.3$. The crossover between the two regimes corresponded to a bubble velocity low enough that the time taken for the bubble walls to merge on collision was longer than the time light took to cross the bubble. We have not been able to probe this velocity region, and cannot rule out the possibility that for very slowly-moving bubble walls, defect suppression due to phase equilibration will outstrip the effect of more bubbles being nucleated, and so lead to fewer defects per unit volume.

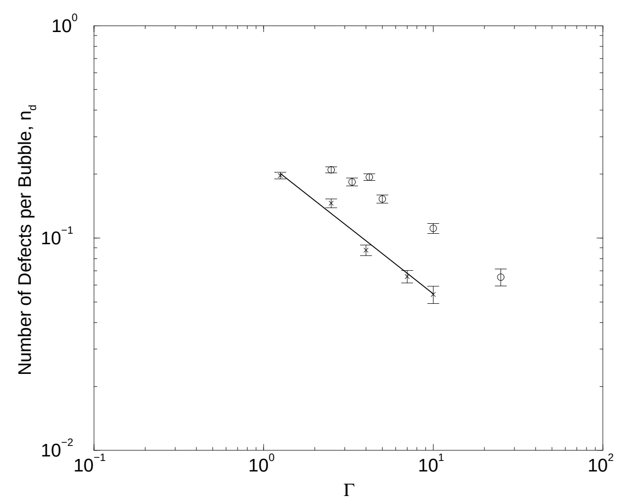

FIG. 5. Log-log plot of number of defects formed per bubble nucleated n_d vs. the friction coefficient Γ. The lower set of points (data points marked with a cross) are the results of this investigation, the upper set (marked with a circle) are those obtained by Ferrera [23]. The two hypotheses of [3] are demonstrably supported – namely i) that n_d should decrease with increasing Γ, and ii) that for a given value of Γ, n_d should be lower for $e = 0.25$ (the gauge symmetry) than for $e = 0$ the global symmetry. The solid line corresponds to the least-squares, best-fit for $n_d \propto \Gamma^{\alpha}$, with $\alpha = -0.65 \pm 0.04$.

A. Results for $\Gamma = 0$

Not shown on Figure 5 (which is drawn with log-log axes) is the result for $\Gamma = 0$ – for this we obtained

$n_d = 0.240 \pm 0.02$. This agrees well with the prediction of the Kibble mechanism, that on purely geometrical grounds the number of defects formed per bubble should be 0.25. As stated in the Introduction, Melfo and Perivolaropoulos [18] found that phase equilibration due to non-simultaneous three-bubble collisions did *not* significantly alter the defect formation rate for global symmetries ($e = 0$), with bubbles moving at the speed of light ($\Gamma = 0$).

We can now confirm that in our gauge-theory model, with bubbles moving at the speed of light, we find no significant deviation from the Kibble mechanism prediction. This result, we feel, is significant enough to be stated in its own section.

VIII. SUMMARY

In this paper we have given an account of our attempts to simulate a first-order phase transition incorporating one of the effects of an early-universe environment – a retarded bubble wall. We have found, as we predicted, that the number of topological defects formed per bubble nucleated, n_d, decreases with the terminal velocity of the bubble wall. Moreover, the number of topological defects produced is significantly lower, for every value of Γ, in the gauge theory than in the global theory.

Slow-moving bubbles allow, in general, more time between collisions and so facilitate the suppression of defect formation due to phase equilibration. Another effect of slow-moving bubbles, however, is the need for more bubbles to be nucleated in order for the phase transition to complete. Our studies show that the surfeit of bubbles dominates the suppression of defects, and that slow-moving bubbles ($0.05 \lesssim v_{\rm ter}/c \leq 1$) actually lead to more defects in a given volume.

This investigation is far from complete. We would have liked to have been able to probe the region $v_{\rm ter} < 0.01$, as this may be more realistic for cosmological scenarios. We would also like to have investigated the effect of plasma conductivity on the defect formation rates, both separately from and in tandem with the probe on the effect of wall terminal velocity. Unfortunately, time and computational constraints did not allow this up until now.

Of course, any such cosmological phase transition would take place not in a static background, but in an expanding universe. The expansion of the Universe has not been taken into account here, but could be extremely important for any high-energy (e.g. GUT-scale) phase transition. This, along with the low-velocity and high-conductivity regimes, is something we hope to probe in the future.

IX. ACKNOWLEDGEMENTS

We would like to thank A.C. Davis, S. Gratton, W. Perkins, P. Saffin, P. Shellard and T. Wiseman for helpful comments and conversations. Computer facilities were provided by the UK National Cosmology Supercomputing Centre in cooperation with Silicon Graphics/Cray Research, supported by HEFCE and PPARC. This work was supported in part by PPARC and an ESF network grant. We would also like to thank CSIC in Madrid, where this project was begun, for hospitality. Support for M.L. was provided by a PPARC studentship and Fitzwilliam College, Cambridge.

[1] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).
[2] A. Ferrera & A. Melfo, Phys. Rev. D **53**, 6852 (1996).
[3] A.C. Davis & M. Lilley, Phys. Rev. D **61**, 043502 (2000).
[4] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan & S. Coleman, *ibid.* **16**, 1762 (1977).
[5] Ya.B. Zel'dovich, I.Yu. Kobzarev & L.B. Okun', Sov. Phys. JETP **40**, 1 (1975).
[6] Ya.B. Zel'dovich & M.Yu. Khlopov, Phys. Lett. B **79**, 239 (1978).
[7] J. Preskill, Phys. Rev. Lett. **43**, 1365 (1979).
[8] U.-L. Pen, U. Seljak & N. Turok, Phys. Rev. Lett. **79**, 1611 (1997).
[9] P. de Bernardis *et. al*, Nature **404**, 955 (2000).
[10] S. Hanany *et al.* Ap. J. **545**, L5 (2000).
[11] A. Vilenkin & E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, (Cambridge University Press, Cambridge 1994).
[12] M.B. Hindmarsh & T.W.B. Kibble, Rept. Prog. Phys. **58**, 477 (1995).
[13] R.H. Brandenberger, A.-C. Davis & A.M. Matheson, Phys. Lett B **218**, 304 (1988).
[14] W.B. Perkins, L. Perivolaropoulos, A.-C. Davis, R.H. Brandenberger & A. Matheson, Nucl. Phys. B **353**, 237 (1991).
[15] P. Bhattacharjee, Phys. Rev. D **40**, 3968 (1989).
[16] V. Berezinsky, B. Hnatyk & A. Vilenkin, astro-ph/0001213.
[17] M.A. Bucher & D. Spergel, Phys. Rev. D **60**, 043505 (1999).
[18] A. Melfo & L. Perivolaropoulos, Phys. Rev. D **52**, 992 (1995).
[19] H. Kurki-Suonio & M. Laine, Phys. Rev. Lett. **77**, 3951 (1996).
[20] M. Dine, R.G. Leigh, P. Huet, A.D. Linde & D. Linde, Phys. Rev. D **46**, 550 (1992); N. Turok, Phys. Rev. Lett. **68**, 1803 (1992); S. Yu. Khlebnikov, Phys. Rev. D **46**, 3223 (1992); G.D. Moore & T. Prokopec, Phys. Rev. Lett. **75**, 777 (1995).
[21] P. John & M.G. Schmidt, Nuc. Phys. B **598**, 291 (2001).

[22] T.W.B. Kibble & A. Vilenkin, Phys. Rev. D **49**, 679 (1995).
[23] A. Ferrera, Phys. Rev. D **57**, 7130 (1998).
[24] M.J. Lilley, `hep-ph/0001241`, *Proceedings of COSMO-99, Trieste, 27 September - 2 October 1999* (World Scientific, Singapore 2000).
[25] T. Vachaspati & A. Vilenkin, Phys. Rev. D **30**, 2036 (1984).
[26] W.H. Press, S.A. Teukolsky, W.T. Vetterling & B.P. Flannery, *Numerical Recipes in Fortran 77* (Cambridge University Press, Cambridge 1992).
[27] E.J. Copeland, P.M. Saffin & O. Törnkvist, Phys. Rev. D **61**, 105005 (2000).
[28] M. Hindmarsh, A.C. Davis & R. Brandenberger, Phys. Rev. D **49**, 1944 (1994).
[29] A.M. Srivastava, Phys. Rev. D **46**, 1353 (1992).
[30] A. Heckler, Phys. Rev. D **51**, 405 (1995).
[31] B.H. Liu, L. McLerran & N. Turok, Phys. Rev. D **46**, 2668 (1992).
[32] J. Ignatius, K. Kajantie, H. Kurki-Suonio & M. Laine, Phys. Rev. D **49**, 3854 (1994).
[33] K.J.M. Moriarty, E. Myers & C. Rebbi, Phys. Lett. B **207**, 411 (1988).
[34] K. Kajantie, M. Karjalainen, M. Laine, J. Peisa & A. Rajantie, Phys. Lett. B **428**, 334 (1998).